

Perturbative BPS-algebras in superstring theory.

C.D.D. Neumann

*Department of Mathematics
University of Amsterdam,
Plantage Muidergracht 24, Amsterdam
Email: `neumann@wins.uva.nl`*

ABSTRACT

This paper investigates the algebraic structure that exists on perturbative BPS-states in the superstring, compactified on the product of a circle and a Calabi-Yau fourfold. This structure was defined in a recent article by Harvey and Moore. It is shown that for a toroidal compactification this algebra is related to a generalized Kac-Moody algebra. The BPS-algebra itself is not a Lie-algebra. However, it turns out to be possible to construct a Lie-algebra with the same graded dimensions, in terms of a half-twisted model. The dimensions of these algebras are related to the elliptic genus of the transverse part of the string algebra. Finally, the construction is applied to an orbifold compactification of the superstring.

1 Introduction.

This paper investigates the algebraic structure that exists on perturbative BPS-states in the superstring, and its relation to generalized Kac-Moody algebras (GKM's). This algebra structure was defined by Harvey and Moore in [15], where they showed that in the toroidal compactification of the heterotic string, this BPS-algebra is closely related to a GKM-algebra. This Lie-algebra actually comes from the left-moving part of the BPS-vertex operators.

In this article I show that this structure also exists on the left-moving part of a toroidal compactified superstring. To this end, I start by reviewing the construction of this left-moving part in the RNS-formulation, which is best suited to compute OPE's, in terms of vertex algebras. I carefully avoid the introduction of operators with fractional spin (parafermions), by splitting the algebra into a transverse part, and a part that combines ghosts and light-cone coordinates. This turns out to be useful for generalizations. I show that after GSO-projection this algebra becomes a vertex super algebra (combining all ghostpictures). By a suitable choice of grading (ghostnumber minus ghostpicture), Lian and Zuckerman's dot and bracket products [13] define a homotopy Gerstenhaber algebra on this space, and then in cohomology a Lie-algebra. Modding out the ghostpicture-equivalence gives a GKM-algebra which is graded by the lightcone momentum lattice.

The graded dimensions of this algebra can be expressed in terms of the (chiral) elliptic genus of the transverse algebra. The GKM-algebra is, however, not graded by the $U(1)$ gradation which is present in the transverse algebra (so that these algebras cannot be identified with algebras like those constructed by Gritsenko and Nikulin [20], which do seem to have such a gradation).

Then the relation between this algebra and the full BPS-algebra of the superstring, compactified on $S_1 \times C_4$ for a Calabi-Yau fourfold C_4 is investigated. Under the assumption that the right-moving momentum has no component in the C_4 direction, the algebra closes without the use of Lorentz-boosts, as introduced by Harvey and Moore. For the torus compactification, the algebra structure is the tensor product of a Lie-algebra on the left-movers and a Lie-algebra on the right movers (and thus not a Lie-algebra). The number of BPS-states can be counted essentially by the elliptic genus of C_4 .

It is then pointed out that there does exist a Lie-algebra on a space with

the same graded dimensions, defined in terms of a half-twisted model [17], where essentially the algebra on the right moving groundstates is replaced by the associative algebra on the chiral ring.

In the last section I look at these structures in a somewhat more complex model, where the transverse algebra is replaced by the sigma-model of the torus orbifold T_8/\mathbf{Z}_2 . The chiral-orbifold resembles in many ways the construction of the monster Lie algebra [18], which can be viewed as the Lie algebra on states of a chiral bosonic string compactified on $T_2 \times (T_{24}/\mathbf{Z}_2)$, where T_{24} is defined by the Leech-lattice.

2 Vertex algebras.

In this section I give definitions of the types of vertex algebras that I will use as building blocks for the superstring algebras. They are vertex (super)-algebras, (I will sometimes omit the word super), and $\mathbf{Z}_2 \times \mathbf{Z}_2$ -graded vertex algebras, which would be called \mathbf{Z}_2^3 -graded vertex para-algebras in the terminology of [10]. Next, I review a general method to construct such algebras from lattices.

2.1 Formal definitions.

A vertex algebra is defined as follows [4, 10, 19]. Let V be a vector space over the complex numbers \mathbf{C} , which is \mathbf{Z}_2 -graded, $V = V^0 \oplus V^1$. Elements in V^0 will be called bosonic, elements in V^1 fermionic. For every $v \in V$, introduce an operator valued function $v(z)$, such that $(\lambda v + \mu w)(z) = \lambda v(z) + \mu w(z)$ for all $v, w \in V$ and $\lambda, \mu \in \mathbf{C}$. These vertex-operators may be viewed as generating functions

$$v(z) = \sum_{n \in \mathbf{Z}} \{v\}_n z^{-n-1} \quad (2.1)$$

for the linear transformations $\{v\}_k : V \rightarrow V$. These transformations must satisfy

$$\{v\}_k w \in V^{a+b} \quad (2.2)$$

for every $v \in V^a, w \in V^b$, and $\{v\}_k w = 0$ for k sufficiently large. The main identity that the operators must satisfy is the so called Jacobi identity. Let $C_i(r, c)$ be a circle in the ζ_i -plane of radius r centered at c , and let f be any polynomial in $\zeta_1, \zeta_1^{-1}, \zeta_2, \zeta_2^{-1}, (\zeta_1 - \zeta_2)^{-1}$. Also let $v \in V^a, w \in V^b$.

Then the Jacobi identity states that for $0 < r_1 < r_2 < r_3$ and $0 < \epsilon < \min(r_3 - r_2, r_2 - r_1)$

$$\begin{aligned}
& \oint_{C_2(r_2,0)} \oint_{C_1(r_3,0)} v(\zeta_1)w(\zeta_2)fd\zeta_1d\zeta_2 \\
& -(-1)^{ab} \oint_{C_2(r_2,0)} \oint_{C_1(r_1,0)} w(\zeta_2)v(\zeta_1)fd\zeta_1d\zeta_2 \\
& = \oint_{C_2(r_2,0)} \oint_{C_1(\epsilon,\zeta_2)} (v(\zeta_1 - \zeta_2)w)(\zeta_2)fd\zeta_1d\zeta_2
\end{aligned} \tag{2.3}$$

This equality is to be interpreted as an equality of matrix elements.

The definition of a $\mathbf{Z}_2 \times \mathbf{Z}_2$ graded vertex algebra is a generalization of the above [10]. In this case, V is graded not only by \mathbf{Z}_2 , but also by $\Gamma = \mathbf{Z}_2 \times \mathbf{Z}_2$, whose elements will be written as $\{00, 01, 10, 11\}$. So

$$V^a = \bigoplus_{\gamma \in \Gamma} V_\gamma^a \tag{2.4}$$

I will also use the notation V_γ for $V_\gamma^0 \oplus V_\gamma^1$. The vertex-operators $v(z)$ do no longer in general have an expansion in integral powers of z . In fact, define

$$\Delta(\gamma_1, \gamma_2) = \begin{cases} 0 & \text{if } \gamma_1 = 00 \text{ or } \gamma_2 = 00 \text{ or } \gamma_1 = \gamma_2 \\ \frac{1}{2} & \text{otherwise} \end{cases} \tag{2.5}$$

which is a bilinear symmetric function mod \mathbf{Z} . Then if $v \in V_{\gamma_1}^a$ and $w \in V_{\gamma_2}^b$, one has

$$v(z)w = \sum_{n \in \mathbf{Z} + \Delta(\gamma_1, \gamma_2)} \{v\}_n w z^{-n-1} \tag{2.6}$$

$$\{v\}_n w \in V_{\gamma_1 + \gamma_2}^{a+b+2\Delta(\gamma_1, \gamma_2)} \tag{2.7}$$

The Jacobi identity must be modified as follows. Let $v \in V_{\gamma_1}^a$, $w \in V_{\gamma_2}^b$ and $u \in V_{\gamma_3}$. Furthermore, let

$$f(\zeta_1, \zeta_2) \in \zeta_1^{\Delta(\gamma_1, \gamma_3)} \zeta_2^{\Delta(\gamma_2, \gamma_3)} (\zeta_1 - \zeta_2)^{\Delta(\gamma_1, \gamma_2)} \mathbf{C}[\zeta_1, \zeta_1^{-1}, \zeta_2, \zeta_2^{-1}, (\zeta_1 - \zeta_2)^{-1}] \tag{2.8}$$

Then under the same conditions for the contours

$$\begin{aligned}
& \oint_{C_2(r_2,0)} \oint_{C_1(r_3,0)} v(\zeta_1)w(\zeta_2)ufd\zeta_1d\zeta_2 \\
& -\eta(\gamma_1, \gamma_2)(-1)^{ab} \oint_{C_2(r_2,0)} \oint_{C_1(r_1,0)} w(\zeta_2)v(\zeta_1)ufd\zeta_1d\zeta_2 \\
& = \oint_{C_2(r_2,0)} \oint_{C_1(\epsilon,\zeta_2)} (v(\zeta_1 - \zeta_2)w)(\zeta_2)ufd\zeta_1d\zeta_2
\end{aligned} \tag{2.9}$$

where $\eta(\gamma_1, \gamma_2)$ is a sign defined by the following table (a choice $x = \pm 1$ must be made)

η	00	01	10	11
00	1	1	1	1
01	1	1	x	-x
10	1	-x	1	x
11	1	x	-x	1

(2.10)

This ends the definition of the $\mathbf{Z}_2 \times \mathbf{Z}_2$ -graded vertex algebra. Note that the subalgebras $V_{00} \oplus V_{01}$, $V_{00} \oplus V_{10}$ and $V_{00} \oplus V_{11}$ all satisfy the definitions of a vertex super-algebra.

The vertex algebras that I will consider all have two distinguished vectors $\mathbf{1}, \omega \in V$ with the following properties. $\mathbf{1}(z)$ is the identity operator on V . The components of $\omega(z)$, defined by

$$\omega(z) = \sum_{n \in \mathbf{Z}} L_n z^{-n-2} \quad (2.11)$$

(so $L_n = \{\omega\}_{n+1}$) generate the Virasoro algebra.

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12} c \delta_{m+n} \quad (2.12)$$

c is called the rank of V . The operator L_0 defines a gradation of V

$$V = \bigoplus_n (V)_n, \quad (V)_n = \{v \in V \mid L_0 v = n v\} \quad (2.13)$$

A vector $v \in (V)_n$ will be called homogeneous of (conformal) weight n , and for such a vector I use the notation $\text{wt}(v) = n$. In this case

$$[L_{-1}, v(z)] = (d/dz)v(z) = \partial v(z) \quad (2.14)$$

$$[L_0, v(z)] = [z(d/dz) + \text{wt}(v)]v(z) \quad (2.15)$$

For the components of $v(z)$ this implies

$$[L_0, \{v\}_k] = (\text{wt}(v) - k - 1)\{v\}_k \quad (2.16)$$

This suggests the introduction of another notation for these components, as follows

$$v_k = \{v\}_{\text{wt}(v)+k-1} \quad (2.17)$$

because one then has the following simple relation for two homogeneous vectors $v, w \in V$

$$\text{wt}(v_k w) = \text{wt}(w) - k \quad (2.18)$$

The vertex operator $v(z)$ is now expanded as follows

$$v(z) = \sum_k v_k z^{-k - \text{wt}(v)} \quad (2.19)$$

2.2 Lattice construction.

A well known way to construct a vertex algebra is the lattice construction [4]. The input is a lattice Λ of dimension n . Let e^1, \dots, e^n be a \mathbf{Z} -basis for the lattice. Then the lattice is characterized by its matrix of inner-products $G^{ij} = \langle e^i, e^j \rangle$ (also known as Gramm-matrix [5]). I assume this matrix to be non-degenerate, and also that $\langle \alpha, \alpha \rangle \in \mathbf{Z}$ for all $\alpha \in \Lambda$. (The polarization formula then shows that the general inner product is half integral). Since G is symmetric, there is a basis of the \mathbf{R} -span of the e^i , call it f^1, \dots, f^n , in which the inner-product is diagonal, $\langle f^i, f^j \rangle = g^{ij}$ where $g = \text{diag}((-1)^k, (+1)^l)$, and (k, l) is the signature of the lattice. Now introduce bosonic fields $\partial\alpha^i(z)$ where $1 \leq i \leq n$, with components defined by

$$\partial\alpha^i(z) = \sum_{k \in \mathbf{Z}} \alpha_k^i z^{-k-1} \quad (2.20)$$

The components of these operators have commutation relations

$$[\alpha_k^i, \alpha_l^j] = k \langle f^i, f^j \rangle \delta_{k+l} \quad (2.21)$$

Next, introduce invertible operators \hat{e}^i where $1 \leq i \leq n$, satisfying

$$[\alpha_k^i, \hat{e}^j] = \delta_{k,0} \langle f^i, e^j \rangle \hat{e}^j \quad (2.22)$$

$$\hat{e}^i \hat{e}^j = \kappa^{c(e^i, e^j)} \hat{e}^j \hat{e}^i \quad (2.23)$$

where κ is some primitive 4-th root of unity, and $c(e^i, e^j)$ is a bilinear function from $\Lambda \times \Lambda$ to \mathbf{Z}_4 , which is antisymmetric, and for which $c(\alpha, \alpha) = 0$ for all $\alpha \in \Lambda$. Now it is possible to define operators for every $\alpha \in \Lambda$ as follows. Let $\alpha = \sum g_i e^i = \sum h_i f^i$. Then

$$\partial\alpha(z) = \sum h_i \partial\alpha^i(z) \quad (2.24)$$

$$\hat{e}_\alpha = (\hat{e}^1)^{g_1} \dots (\hat{e}^n)^{g_n} \quad (2.25)$$

Now since the two sets of operators $X = \{\hat{e}^i, \alpha_0^i\}$ and $Y = \{\alpha_k^i\}$ (where $k \neq 0$) mutually commute, a representation for the algebra can be written as a tensor product of representations for X and Y . A representation for X (the lattice algebra) can be constructed by introducing a vacuum $|0\rangle$, and then for every $\alpha \in \Lambda$ a vector

$$|\alpha\rangle = \hat{e}_\alpha |0\rangle \quad (2.26)$$

Then the operators in X act on this as

$$\alpha_0^i |\beta\rangle = \langle f^i, \beta \rangle |\beta\rangle \quad (2.27)$$

$$\hat{e}_\alpha |\beta\rangle = e^{i\phi} |\alpha + \beta\rangle \quad (2.28)$$

for some phase $\phi(\alpha, \beta)$. I will call this representation-space $\hat{\Lambda}$. The algebra Y can be viewed as n copies of a Heisenberg algebra. A representation for this algebra can be constructed by introducing a vacuum-vector $\mathbf{1}$, and then requiring that

$$\alpha_k^i \mathbf{1} = 0, \quad \text{for } k > 0 \quad (2.29)$$

The representation space for one copy of the Heisenberg algebra will be called H . Consequently, V has the form

$$V = \hat{\Lambda} \otimes H^n \quad (2.30)$$

and this space will be called $V(\Lambda)$. (In the sequel, I will use this notation even if Λ is not a lattice, but for example a subset of points from a lattice). A general vector in $V(\Lambda)$ has the form

$$\Psi = \alpha_{-k_1-1}^{i_1} \dots \alpha_{-k_m-1}^{i_m} |\beta\rangle \quad (2.31)$$

where the $k_i \geq 0$. To it, one can associate the following vertex operator

$$\Psi(z) = e_\beta z^{\beta_0} : \left[\frac{d^{k_1}}{k_1! dz^{k_1}} \partial \alpha^{i_1}(z) \right] \dots \left[\frac{d^{k_m}}{k_m! dz^{k_m}} \partial \alpha^{i_m}(z) \right] \exp \left(- \sum_{k \neq 0} \frac{\beta_k z^{-k}}{k} \right) : \quad (2.32)$$

The normal ordering is defined in the usual way

$$:\alpha_k^i \alpha_l^j := \begin{cases} \alpha_k^i \alpha_l^j & \text{if } k < l \\ \alpha_l^j \alpha_k^i & \text{otherwise} \end{cases} \quad (2.33)$$

The vertex operator can also be written symbolically as

$$\Psi(z) =: \left[\frac{d^{k_1}}{k_1! dz^{k_1}} \partial \alpha^{i_1}(z) \right] \cdots \left[\frac{d^{k_m}}{k_m! dz^{k_m}} \partial \alpha^{i_m}(z) \right] e^{\beta(z)} : \quad (2.34)$$

which is more usual in string theory. With these definitions, $V(\Lambda)$ becomes a vertex para-algebra, if the cocycle $c(e^i, e^j)$ is chosen properly. It is easy to define the \mathbf{Z}_2 grading in this algebra. Namely, let

$$\Lambda^a = \{\alpha \in \Lambda \mid \langle \alpha, \alpha \rangle \in 2\mathbf{Z} + a\} \quad (2.35)$$

be the subsets of points in Λ of even and odd lengths. Then $V^a = V(\Lambda^a)$. It is a well known fact that if Λ is integral, $V(\Lambda)$ is a vertex super-algebra. It will be a $\mathbf{Z}_2 \times \mathbf{Z}_2$ graded vertex algebra iff there are exactly two (and not more) basisvectors among the e^i , say e^p and e^q , for which $\langle e^p, e^q \rangle \in \mathbf{Z} + \frac{1}{2}$. The $\mathbf{Z}_2 \times \mathbf{Z}_2$ grading is then defined as follows. Let Λ^+ be the lattice that is the \mathbf{Z} -span of the e^i , but with e^p and e^q replaced by $2e^p, 2e^q$. (This sublattice will be integral). Then define the following four vectors

$$v_{00} = 0, v_{01} = e^p, v_{10} = e^q, v_{11} = e^p + e^q \quad (2.36)$$

Then Λ can be written as the union of four translates of Λ^+ , and this defines the gradation

$$V_\gamma = V(\Lambda^+ + v_\gamma) \quad (2.37)$$

The algebra contains the two special vectors, the identity $\mathbf{1} = |0\rangle$, and the Virasoro element. This element can be written as

$$\omega = \frac{1}{2} \sum_{i,j} g_{ij} \alpha_{-1}^i \alpha_{-1}^j |0\rangle \quad (2.38)$$

This definition gives $\alpha^i(z)$ conformal weight 1, $\text{rank}(V) = c = n$, and

$$L_0 |\beta\rangle = \frac{\langle \beta, \beta \rangle}{2} |\beta\rangle \quad (2.39)$$

I will call ω as described here the standard Virasoro element. Using the tensor product structure of V , it is not hard to derive the following formula for the partition function

$$\text{Tr}_V q^{L_0 - c/24} = \frac{\sum_{\alpha \in \Lambda} q^{\langle \alpha, \alpha \rangle / 2}}{q^{c/24} \prod_{k=1}^{\infty} (1 - q^k)^c} = \frac{\theta_\Lambda(q)}{\eta(q)^n} \quad (2.40)$$

Note that this function is well defined only if the lattice is positive definite. The choice of $\omega(z)$ as described above is not the only one that is possible. For example, for every $\alpha \in \Lambda$, the operator $\omega'(z) = \omega(z) + \partial^2 \alpha(z)$ also satisfies the definition. This possibility is used in the construction of the ghost-sector of the superstring. With this definition, the $\alpha^i(z)$ still have conformal weight one, but

$$L'_0|\beta\rangle = \frac{\langle\beta - 2\alpha, \beta\rangle}{2}|\beta\rangle \quad (2.41)$$

2.3 Tensor product algebras.

Suppose $V(\Lambda_1), V(\Lambda_2)$ are both vertex algebras. One of them might be $\mathbf{Z}_2 \times \mathbf{Z}_2$ -graded. Then their tensor product $V(\Lambda_1) \otimes V(\Lambda_2)$ can be given the structure of a vertex algebra, by saying

$$V(\Lambda_1) \otimes V(\Lambda_2) = V(\Lambda_1 \otimes \Lambda_2) \quad (2.42)$$

Here by $\Lambda_1 \otimes \Lambda_2$ I mean the orthogonal product of the two lattices. The effect is that the bosonic fields used in the construction of the two factors mutually commute. However, care must be taken for the operators \hat{e}_α . The correct prescription is that if $\alpha \in \Lambda_1$ and $\beta \in \Lambda_2$, then one should have

$$\hat{e}_\alpha \hat{e}_\beta = (-1)^{\langle\alpha, \alpha\rangle \langle\beta, \beta\rangle} \hat{e}_\beta \hat{e}_\alpha \quad (2.43)$$

3 Toroidal compactified superstring.

In this section I give a description of the left moving algebra of the superstring, compactified on the unique Lorentzian lattice in ten dimensions E_{10} , in the RNS formulation, in terms of vertex algebras. I will choose a basis for the coordinates of the string in which E_{10} splits into an orthogonal product $E_8 \otimes E_{1,1}$, and then describe the coordinates compactified on E_8 (the transverse part) and those on $E_{1,1}$ (the lightcone) separately.

3.1 The transverse part.

I will restrict my attention to the left movers. The bosonic fields, compactified on E_8 , are described by a vertex-algebra $V(E_8)$. I will call the bosonic

fields used in the construction $\partial X^i(z)$ where $1 \leq i \leq 8$. They satisfy the OPE's

$$\partial X^i(z) \partial X^j(w) = \frac{\delta^{i,j}}{(z-w)^2} + \dots \quad (3.1)$$

The fermionic fields are described by a $\mathbf{Z}_2 \times \mathbf{Z}_2$ -graded vertex algebra $V(D_4^*)$ (before GSO-projection) [8, 10, 11]. Here D_4^* is the weight-lattice of the Lie-algebra D_4 , the dual of the root lattice. Let me relate this compact definition to the concepts from string theory. First I will define the $\mathbf{Z}_2 \times \mathbf{Z}_2$ gradation on $V(D_4^*)$. A basis for the lattice D_4 is given by

$$e^1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, e^2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, e^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, e^4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad (3.2)$$

(in terms of the basis f^i , with $\langle f^i, f^j \rangle = \delta^{i,j}$) and the Gramm-matrix then becomes (giving actually the Cartan matrix for the Lie-algebra D_4)

$$\langle e^i, e^j \rangle = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \quad (3.3)$$

The dual of D_4 can be written as a union of four translates of D_4 [5]

$$D_4^* = \bigcup_{\gamma \in \mathbf{Z}_2 \times \mathbf{Z}_2} (D_4 + \Delta_\gamma) \quad (3.4)$$

where the translation vectors are given by

$$\Delta_{00} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \Delta_{01} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \Delta_{10} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \Delta_{11} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \quad (3.5)$$

The translation vectors are labeled by elements of the group $\mathbf{Z}_2 \times \mathbf{Z}_2$ because $\Delta_{\gamma_1} + \Delta_{\gamma_2} = \Delta_{\gamma_1 + \gamma_2}$ modulo elements of D_4 . (They are actually representants of the quotient D_4^*/D_4). This defines the $\mathbf{Z}_2 \times \mathbf{Z}_2$ -gradation of $V(D_4^*)$

$$V(D_4^*) = \bigoplus_{\gamma \in \mathbf{Z}_2 \times \mathbf{Z}_2} \hat{\Lambda}_{D_4 + \Delta_\gamma} \otimes H^4 = \bigoplus_{\gamma \in \mathbf{Z}_2 \times \mathbf{Z}_2} V_\gamma \quad (3.6)$$

The subspace $V_{00} \oplus V_{01}$ is the vertex super-algebra $V(\mathbf{Z}^4)$. It is a well known result of boson-fermion correspondence theorems [6] that this algebra is isomorphic to the algebra constructed from eight fermionic fields with NS-boundary conditions. The correspondence is as follows. The vectors $\beta^i = f^i$, with $1 \leq i \leq 4$ form a basis for this lattice \mathbf{Z}^4 . Then the vertex operators defined by

$$\psi^i(z) =: e^{\beta^i(z)} :, \quad \bar{\psi}^i(z) =: e^{-\beta^i(z)} : \quad (3.7)$$

are complex fermions. They satisfy OPE's

$$\psi^i(z) \bar{\psi}^j(w) = \frac{\delta^{i,j}}{(z-w)} + \cdots \quad (3.8)$$

For later use, I will define real fermions in terms of these

$$b^{2i-1}(z) = \frac{\psi^i(z) + \bar{\psi}^i(z)}{\sqrt{2}}, \quad b^{2i}(z) = \frac{\psi^i(z) - \bar{\psi}^i(z)}{i\sqrt{2}} \quad (3.9)$$

which satisfy

$$b^i(z) b^j(w) = \frac{\delta^{i,j}}{(z-w)} + \cdots \quad (3.10)$$

So the space $V_{00} \oplus V_{01}$ can be identified as the NS-sector of the string theory. Now one can check that the operators $\psi^i, \bar{\psi}^i$ have an expansion in half integral powers of z when acting on V_{10} or V_{11} . So $V_{10} \oplus V_{11}$ can be identified as the R-sector. The operators $\hat{e}_{\Delta_{10}}$ and $\hat{e}_{\Delta_{11}}$ are examples of spectral flow operators, they are invertible operators that map NS-states to R-states and vice versa. The tensor product of the fermionic and bosonic algebras are a $\mathbf{Z}_2 \times \mathbf{Z}_2$ -graded vertex algebra, and I define

$$V_\gamma^T = V(E_8) \otimes V(D_4 + \Delta_\gamma) \quad (3.11)$$

Note that this division into a NS- and a R-sector depends upon the choice which element V_γ with $\gamma \in \{01, 10, 11\}$ is added to V_{00} to form a vertex super-algebra. The fermions in this algebra still have NS-boundary conditions when acting on $V_{00} \oplus V_\gamma$, and R-boundary conditions when acting on the remaining two V 's. In the algebra $V(D_4^*)$, this is of no consequence, because the three possible choices are related by triality [10], but in general it does matter.

3.2 Partition functions.

It is easy to obtain the partition functions for the algebras constructed in the previous section. For the bosonic part one finds

$$\mathrm{Tr}_{V(E_8)} q^{L_0-1/3} = \frac{\theta_{E_8}(q)}{\eta^8(q)} = \frac{E_4(q)}{\eta^8(q)} = q^{-1/3}(1 + 248q + \dots) \quad (3.12)$$

where $E_4(q)$ is the normalized Eisenstein series of weight 4. As for the fermionic part, these are given by

$$P_\gamma(q) = \mathrm{Tr}_{V(D_4+\Delta_\gamma)} q^{L_0-1/6} = \frac{\theta_{D_4+\Delta_\gamma}(q)}{\eta^4(q)} \quad (3.13)$$

From triality it follows that the P_γ for $\gamma \neq 00$ are all equal. Now from the fermionic formulation of the superstring, or by using some modular identities, one finds the familiar result [12]

$$P_\gamma(q) = 8q^{1/3} \prod_{k=1}^{\infty} (1 + q^k)^8, \quad \text{for } \gamma \neq 00 \quad (3.14)$$

These partition functions can be refined by introducing a second gradation in the algebra $V(D_4^*)$, leading to a chiral version of the elliptic genus. This gradation will be defined by a $U(1)$ subalgebra of the D_4 algebra that acts on $V(D_4^*)$. It requires the choice of a specific weight one vertex-operator. There are three preferred choices that can be made, namely the bosonic fields $J^\gamma(z)$ associated with the lattice points $2\Delta_\gamma \in D_4$, with $\gamma \neq 00$. Their zero modes act like

$$J_0^\gamma |\beta\rangle = 2\langle \Delta_\gamma, \beta \rangle |\beta\rangle \quad (3.15)$$

and they commute with the Heisenberg-algebra. Among these, J_0^{11} is special, because $J^{11}(z)$ is the $U(1)$ current that combines with the $N = 1$ algebra, described in section 4, to give the standard $N = 2$ algebra in the transverse theory. This $U(1)$ current remains well defined in more general compactifications with space-time supersymmetry [9]. Furthermore, $(-1)^{J_0^{11}}$ defines (part of) the GSO-projection. It acts as $+1$ on V_{00}^T, V_{11}^T , and as -1 on V_{01}^T, V_{10}^T . So let me write simply $J_0 = J_0^{11}$. Now one can define the functions

$$P_\gamma(q, y) = \mathrm{Tr}_{V(D_4+\Delta_\gamma)} q^{L_0-1/6} y^{J_0} = \frac{\sum_{\alpha \in D_4+\Delta_\gamma} q^{\langle \alpha, \alpha \rangle / 2} y^{2\langle \Delta_{11}, \alpha \rangle}}{\eta^4(q)} \quad (3.16)$$

which reduces to $P_\gamma(q)$ for $y = 1$. Again, using the fermionic formulation, these functions can be expressed in a product form. The result is

$$P_\gamma(q, y) = \begin{cases} \theta_1^+(q, y) & \text{if } \gamma = 00 \\ \theta_1^-(q, y) & \text{if } \gamma = 11 \\ \theta_2(q, y) & \text{otherwise} \end{cases} \quad (3.17)$$

where

$$\begin{aligned} \theta_1^\pm &= q^{-1/6} \prod_{n=0}^{\infty} (1 + q^{n+\frac{1}{2}} y^{-2}) (1 + q^{n+\frac{1}{2}})^6 (1 + q^{n+\frac{1}{2}} y^2) \\ &\quad \pm q^{-1/6} \prod_{n=0}^{\infty} (1 - q^{n+\frac{1}{2}} y^{-2}) (1 - q^{n+\frac{1}{2}})^6 (1 - q^{n+\frac{1}{2}} y^2) \end{aligned} \quad (3.18)$$

$$\theta_2 = q^{1/3} (4y^{-1} + 4y) \prod_{n=1}^{\infty} (1 + q^n y^{-2}) (1 + q^n)^6 (1 + q^n y^2) \quad (3.19)$$

These functions have the following series expansions

$$\theta_1^+(q, y) = q^{-1/6} (1 + (6y^{-2} + 16 + 6y^2)q + \dots) \quad (3.20)$$

$$\theta_1^-(q, y) = q^{1/3} ((y^{-2} + 6 + y^2) + (16y^{-2} + 32 + 16y^2)q + \dots) \quad (3.21)$$

$$\theta_2(q, y) = q^{1/3} ((4y^{-1} + 4) + (4y^{-3} + 28y^{-1} + 28y + 4y^3)q + \dots) \quad (3.22)$$

By using the spectral flow operator $\hat{e}_{\Delta_{11}}$ one can find the following identity

$$\begin{aligned} P_{\gamma+11}(q, y) &= \text{Tr}_{V(D_4+\Delta_\gamma)} \hat{e}_{\Delta_{11}}^{-1} q^{L_0-1/6} y^{J_0^{11}} \hat{e}_{\Delta_{11}} = \\ &= \text{Tr}_{V(D_4+\Delta_\gamma)} q^{L_0+\frac{1}{2}J_0^{11}+1/3} y^{J_0^{11}+2} = \sqrt{q} y^2 P_\gamma(q, \sqrt{q}y) \end{aligned} \quad (3.23)$$

In general, applying spectral flow n times to a vector with (L_0, J_0) -eigenvalues (q, k) sends it to a vector with eigenvalues $(q + \frac{nk}{2} + \frac{n^2}{2}, k + 2n)$ [16] (iff $c = 12$). So let $(q, -k)$ be a vector in V_{01} . Then k must be odd, and applying spectral flow k times sends it to a vector (q, k) in V_{10} . This leads to the identity

$$P_{01}(q, y) = P_{10}(q, y) \quad (3.24)$$

These spectral flow arguments of course remain valid for more general compactifications. In the GKM-algebras that will be constructed, I will be interested only in the partition functions of V_{01}^T , which will count bosons, and

V_{11}^T , which counts fermions (with given momentum). So one can define a 'supersymmetric index' analogously to [7]

$$I(q, y) = \text{Tr}_{V_{01}^T \oplus V_{11}^T} (-1)^{J_0} q^{L_0-1/2} y^{J_0} = \frac{E_4(q)}{\eta^8(q)} (\theta_1^-(q, y) - \theta_2(q, y)) \quad (3.25)$$

By using (3.24), this function can also be written as a trace over the R-sector, making it look like a chiral version of the elliptic genus

$$I(q, y) = \text{Tr}_{V_{10}^T \oplus V_{11}^T} (-1)^{J_0} q^{L_0-1/2} y^{J_0} \quad (3.26)$$

By using spectral flow it can also be expressed as a trace over the NS-sector

$$I(q, y) = \text{Tr}_{V_{00}^T \oplus V_{01}^T} (-1)^{J_0} q^{L_0+J_0} y^{J_0+2} \quad (3.27)$$

The function turns out to have all the defining properties of a so called weak Jacobi form of index 2 and weight 0 [1, 2, 3]. The space of such forms is two dimensional, with basis

$$J_1 = E_4(q) \left(\frac{\phi_{10,1}(q, y)}{\eta^{24}(q)} \right)^2 = (y^{-2} - 4y^{-1} + 6 - 4y + y^2) + O(q) \quad (3.28)$$

$$J_2 = \left(\frac{\phi_{12,1}(q, y)}{\eta^{24}(q)} \right)^2 = (y^{-2} + 20y^{-1} + 102 + 20y + y^2) + O(q) \quad (3.29)$$

Here $\phi_{10,1}$ and $\phi_{12,1}$ are unique cusp forms of index 1 and weights 10 and 12 respectively, described in [1]. Comparing a few coefficients shows that $I = J_1$.

3.3 The lightcone and ghosts.

In this section I describe the lightcone coordinates of the superstring plus the ghost-sector. I describe them together, because only together do they form proper vertex algebras. I claim that this algebra can be described as (a subalgebra of) the tensorproduct of four vertex algebras, namely $V(E_{1,1}) \otimes V_1(\mathbf{Z}) \otimes V_2(\mathbf{Z}) \otimes V(L)$. I will now describe where these factors come from.

The vertex algebra $V(E_{1,1})$ corresponds to the bosonic coordinates of the superstring, compactified on $E_{1,1}$. This lattice has Gramm-matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.30)$$

I will call the bosonic fields used in the construction $\partial X^0(z), \partial X^9(z)$, and they satisfy

$$\partial X^i(z) \partial X^j(w) = \frac{\eta^{ij}}{(z-w)^2} + \dots \quad (3.31)$$

where η is diagonal, $-\eta^{00} = \eta^{99} = 1$.

The two factors $V_{1,2}(\mathbf{Z})$ describe the b, c -ghostsystem and the η, ξ system respectively, which is part of the β, γ -ghostsystem [8]. Here the lattice \mathbf{Z} is the lattice with basis $e^1 = f^1$, and $\langle f^1, f^1 \rangle = 1$. Now let $\sigma = f^1$ be in the lattice coming from $V_1(\mathbf{Z})$ and $\chi = f^1$ in the lattice coming from $V_2(\mathbf{Z})$. Then

$$c(z) = :e^{\sigma(z)}:, \quad b(z) = :e^{-\sigma(z)}: \quad (3.32)$$

$$\xi(z) = :e^{\chi(z)}:, \quad \eta(z) = :e^{-\chi(z)}: \quad (3.33)$$

The b, c -system satisfies the OPE's (and similar for ξ, η)

$$c(z)b(w) = \frac{1}{(z-w)} + \dots \quad (3.34)$$

The factor $V(L)$ is a $\mathbf{Z}_2 \times \mathbf{Z}_2$ graded vertex algebra. The lattice L , which has signature $(1, 1)$, can be written as the union of four translates of an even sublattice L^+ . A basis for L^+ is given by

$$e^1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, e^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (3.35)$$

in terms of the basis f^1, f^2 , where $\langle f^i, f^j \rangle = \text{diag}(-1, 1)$. Then L is described by

$$L = \bigcup_{\gamma \in \mathbf{Z}_2 \times \mathbf{Z}_2} (L^+ + \delta_\gamma) \quad (3.36)$$

where the translation vectors are given by

$$\delta_{00} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \delta_{01} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \delta_{10} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \delta_{11} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \quad (3.37)$$

This defines the $\mathbf{Z}_2 \times \mathbf{Z}_2$ -gradation of $V(L)$, just as in the case of $V(D_4^*)$

$$V(L) = \bigoplus_{\gamma \in \mathbf{Z}_2 \times \mathbf{Z}_2} V(L^+ + \delta_\gamma) = \bigoplus_{\gamma \in \mathbf{Z}_2 \times \mathbf{Z}_2} V_\gamma \quad (3.38)$$

Clearly, the lattice L contains the points $\phi = f^1$, and $\rho = f^2$. I can now identify

$$b^+(z) =: e^{\rho(z)} :, \quad b^-(z) =: e^{-\rho(z)} : \quad (3.39)$$

$$\gamma(z) =: e^{\phi(z)} \eta(z) :, \quad \beta(z) =: e^{-\phi(z)} \partial \xi(z) : \quad (3.40)$$

The fermionic fields $b^\pm(z)$ satisfy

$$b^+(z)b^-(w) = \frac{1}{(z-w)} + \dots \quad (3.41)$$

They are the lightcone fermions of the matter-system. The bosonic fields $\beta(z), \gamma(z)$ satisfy

$$\gamma(z)\beta(w) = \frac{1}{(z-w)} + \dots \quad (3.42)$$

Note that $:\gamma(z)\beta(z): = -\phi(z)$ and $:c(z)b(z): = \sigma(z)$. For later use I define

$$b^0(z) = \frac{b^+(z) - b^-(z)}{\sqrt{2}}, \quad b^g(z) = \frac{b^+(z) + b^-(z)}{\sqrt{2}} \quad (3.43)$$

In $V(L)$ there is no ambiguity about what to call the NS-sector, and what the R-sector, because only V_{00} and V_{01} contain states without ghost-excitations, so they must be the NS-sector. Also, the entire lightcone plus ghosts algebra is again a $\mathbf{Z}_2 \times \mathbf{Z}_2$ graded vertex algebra, and I define

$$V_\gamma^L = V(E_{1,1}) \otimes V_1(\mathbf{Z}) \otimes V_2(\mathbf{Z}) \otimes V_\gamma \quad (3.44)$$

3.4 Fitting it together.

The NS-sector of the total left moving algebra is of course the product of the NS-sectors of transverse part and the lightcone plus ghost part, and similar for the R-sector. Thus

$$V_{NS} = (V_{00}^L \oplus V_{01}^L) \otimes (V_{00}^T \oplus V_{01}^T) \quad (3.45)$$

$$V_R = (V_{10}^L \oplus V_{11}^L) \otimes (V_{10}^T \oplus V_{11}^T) \quad (3.46)$$

The reader can check that this space again has the structure of a $\mathbf{Z}_2 \times \mathbf{Z}_2$ graded vertex algebra, with a gradation defined by

$$V_{00} = V_{00}^L \otimes V_{00}^T \oplus V_{01}^L \otimes V_{01}^T \quad (3.47)$$

$$V_{01} = V_{00}^L \otimes V_{01}^T \oplus V_{01}^L \otimes V_{00}^T \quad (3.48)$$

$$V_{10} = V_{10}^L \otimes V_{10}^T \oplus V_{11}^L \otimes V_{11}^T \quad (3.49)$$

$$V_{11} = V_{10}^L \otimes V_{11}^T \oplus V_{11}^L \otimes V_{10}^T \quad (3.50)$$

I will call this algebra the superstring vertex-algebra. Then the subspace $V_{00} \oplus V_{10}$ is a vertex super-algebra, and this is exactly the space that is projected out by the GSO-projection [14].

4 The BRST-construction.

In order to define the physical states of the string, one needs to define the BRST-operator Q . This operator is defined in terms of the $N = 1$ supersymmetry algebra that is present in the superstring algebra [12, 14]. The $N = 1$ algebra is generated by two fields $T(z), G(z)$, that must satisfy the following OPE's

$$T(z)T(w) = \frac{c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \dots \quad (4.1)$$

$$T(z)G(w) = \frac{3G(w)/2}{(z-w)^2} + \frac{\partial G(w)}{(z-w)} + \dots \quad (4.2)$$

$$G(z)G(w) = \frac{2c/3}{(z-w)^3} + \frac{2T(w)}{(z-w)} + \dots \quad (4.3)$$

where c is some constant. This structure is present in both the matter sector and the ghost sector. In the matter sector it is given by

$$T_m(z) = \frac{1}{2}\eta_{ij} :(\partial X^i)(\partial X^j) + (\partial b^i)b^j: \quad (4.4)$$

$$G_m(z) = \eta_{ij}(\partial X^i)b^j \quad (4.5)$$

with $c = 15$. $T_m(z)$ is just the operator $\omega(z)$ corresponding to the standard Virasoro element in this part of the algebra. In the ghostsector one has

$$T_g(z) = - :(\partial b)c: - 2 :b\partial c: - \frac{3}{2} : \beta \partial \gamma : - \frac{1}{2} :(\partial \beta)\gamma: \quad (4.6)$$

$$G_g(z) = -2b\gamma + c\partial\beta + \frac{3}{2}(\partial c)\beta \quad (4.7)$$

with $c = -15$. In this case $T_g(z)$ is not equal to $\omega(z)$, but is actually given by

$$T_g(z) = \omega(z) + :\partial(\frac{3}{2}cb - \gamma\beta): \quad (4.8)$$

This gives the fields b, c, β, γ conformal weights $2, -1, \frac{3}{2}, -\frac{1}{2}$ respectively. The operators T_m and T_g are in V_{00} , while the operator G_m and G_g are in V_{01} . The BRST-current $Q(z)$ is defined as

$$Q(z) =:(T_m + \frac{1}{2}T_g)c + (G_m + \frac{1}{2}G_g)\gamma: \quad (4.9)$$

This fermionic field has conformal weight 1. Its zero mode Q_0 , which I will write as Q , satisfies [14]

$$Q^2 = 0 \quad (4.10)$$

Since $Q(z)$ is in V_{00} , it maps every V_γ to itself. To define the cohomology problem, I have to define some subspaces of the superstring algebra. First the factor $V_2(\mathbf{Z}) \otimes V(L)$ in the construction of this algebra is restricted to a subalgebra W , which is graded by the ghostpicture (number) [8]. This subalgebra can be defined as follows. In the product lattice $L \otimes \mathbf{Z}$ one can define a set of points given by $v_k = 2k\delta_{10}$ with $k \in \mathbf{Z}/2$ (These vectors do not have a component in the \mathbf{Z} lattice direction). These points define a set of vectors $|v_k\rangle$ in the algebra $V_2(\mathbf{Z}) \otimes V(L)$. From every $|v_k\rangle$, I can construct a space W_k , by letting the component operators of the fields $b^\pm(z), \beta(z), \gamma(z)$ act on it in all possible ways. The number k will be called the ghostpicture. I now define

$$W = \bigoplus_{k \in \mathbf{Z}/2} W_k \quad (4.11)$$

One can easily see that the subalgebra W is again a $\mathbf{Z}_2 \times \mathbf{Z}_2$ graded vertex algebra, and that the ghostpicture defines a gradation of it. This then defines a subalgebra of the superstring algebra (by using W instead of $V_2(\mathbf{Z}) \otimes V(L)$ in its construction), which is also graded by ghostpicture number. Let me call the graded pieces of this subalgebra $V_{\gamma,k}^s$. Note that $k \in \mathbf{Z} + \Theta(\gamma)$ where

$$\Theta(\gamma) = \begin{cases} 0 & \text{if } \gamma = 00, 01 \text{ (NS sector)} \\ \frac{1}{2} & \text{if } \gamma = 10, 11 \text{ (R-sector)} \end{cases} \quad (4.12)$$

Since Q has ghostpicture number 0, it leaves all $V_{\gamma,k}^s$ invariant. Next, I need another gradation, by ghostnumber (not to be confused with ghostpicture).

This gradation is defined by the zero-mode of the ghostnumber current defined by [14]

$$J(z) =: cb - \gamma\beta := \sigma(z) + \phi(z) \quad (4.13)$$

It gives c, γ ghostnumber 1 and b, β ghostnumber -1 . Consequently Q has ghostnumber 1. In general, the ghostnumber will be in $\mathbf{Z} + \Theta(\gamma)$ for $V_{\gamma,k}^s$. The final gradation that I will use is the gradation by the lightcone-lattice $E_{1,1}$. It will in the end define a gradation on the GKM-algebras that I construct. Let me write

$$V_{\gamma,k}^s = \bigoplus_{\alpha \in E_{1,1}, n \in \mathbf{Z} + \Theta(\gamma)} V_{\gamma,k}^s(\alpha, n) \quad (4.14)$$

where n is the ghostnumber. I am now ready to define the cohomology problem. What I'm interested in is the relative cohomology of Lian and Zuckerman [14]. This is Q -cohomology in a subspace of $V_{\gamma,k}^s$, defined by

$$\ker L_0 \cap \ker b_0 \quad (4.15)$$

in the NS-sector ($\gamma = 00, 01$), and

$$\ker L_0 \cap \ker b_0 \cap \ker G_0 \cap \ker \beta_0 \quad (4.16)$$

in the R-sector ($\gamma = 10, 11$). Here L_0, b_0, G_0, β_0 are the zero-modes of the fields $T_g(z) + T_m(z), b(z), G_g(z) + G_m(z), \beta(z)$ respectively. Note that G_0 and β_0 only exist as operators in the R-sector. Let $C_{\gamma,k}$ be this subspace. The cohomology is

$$H_{\gamma,k}(\alpha, n) = \ker_Q C_{\gamma,k}(\alpha, n) / \text{im}_Q C_{\gamma,k}(\alpha, n-1) \quad (4.17)$$

(This is to be contrasted with the absolute cohomology H^{abs} , which is the Q -cohomology of $V_{\gamma,k}^s$). The results of this calculation are well known [8, 14]. I will state the results. There are two cases to consider, $\alpha = 0$ and $\alpha \neq 0$. For $\alpha \neq 0$, one has

$$H_{\gamma,k}(\alpha, n) = 0, \quad \text{iff } n - k \neq 1 \quad (4.18)$$

$$\dim H_{\gamma,k}(\alpha, k+1) = \dim(V_{\gamma+01}^T) - \frac{\langle \alpha, \alpha \rangle}{2} + \frac{1}{2} \quad (4.19)$$

where $(V_{\gamma}^T)_n$ is the subspace of V_{γ}^T of conformal dimension n . If $\alpha = 0$, one finds

$$H_{\gamma,k}(0, n) = 0, \quad \text{iff } n - k \neq 0, 1, 2 \quad (4.20)$$

$$\dim H_{\gamma,k}(0, k) = \dim H_{\gamma,k}(0, k+2) = \begin{cases} 1 & \text{NS-sector} \\ 0 & \text{R-sector} \end{cases} \quad (4.21)$$

$$\dim H_{\gamma,k}(0, k+1) = \begin{cases} 10 & \gamma = 00 \\ 0 & \gamma = 01 \\ 16 & \gamma = 10 \\ 16 & \gamma = 11 \end{cases} \quad (4.22)$$

After the GSO-projection, which projects out $V_{00} \oplus V_{10}$, the physical states of the superstring are given by

$$H = \left(\bigoplus_{k \in \mathbf{Z}, \alpha \in E_{1,1}} H_{00,k}(\alpha, k+1) \right) \oplus \left(\bigoplus_{k \in \mathbf{Z} + \frac{1}{2}, \alpha \in E_{1,1}} H_{10,k}(\alpha, k+1) \right) \quad (4.23)$$

modulo the ghostpicture equivalence [8], to be described in the next section. Note that representatives of $H_{00,k}(\alpha, k+1)$ are fermionic states (in the vertex algebra) and representatives of $H_{10,k}(\alpha, k+1)$ are bosonic.

5 Gerstenhaber algebra.

In this section I describe the Gerstenhaber-algebra which is present in the absolute superstring cohomology. Let me recall the definition of such an algebra [13]. It is a $\mathbf{Z}_2 \times \mathbf{Z}$ -graded vector space V

$$V = \bigoplus_{a \in \mathbf{Z}_2, m \in \mathbf{Z}} V_m^a \quad (5.1)$$

equipped with two multiplication operations, the dot product

$$\cdot : V_m^a \times V_n^b \rightarrow V_{m+n}^{a+b} \quad (5.2)$$

and the bracket

$$\{, \} : V_m^a \times V_n^b \rightarrow V_{m+n-1}^{a+b-1} \quad (5.3)$$

Let $u \in V_m^a, v \in V_n^b, w \in V_p^c$. The dot product is associative commutative

$$u \cdot v = (-1)^{ab} v \cdot u \quad (5.4)$$

$$(u \cdot v) \cdot w = u \cdot (v \cdot w) \quad (5.5)$$

The bracket satisfies

$$\{u, v\} = -(-1)^{(a-1)(b-1)}\{v, u\} \quad (5.6)$$

$$\begin{aligned} & (-1)^{(a-1)(c-1)}\{u, \{v, w\}\} + (-1)^{(c-1)(b-1)}\{w, \{u, v\}\} \\ & + (-1)^{(b-1)(a-1)}\{v, \{w, u\}\} = 0 \end{aligned} \quad (5.7)$$

(It is like a Lie-bracket with the definitions of fermions and bosons swapped). Finally there is a relation between the bracket and the dot product

$$\{u, v \cdot w\} = \{u, v\} \cdot w + (-1)^{(a-1)b}v \cdot \{u, w\} \quad (5.8)$$

5.1 The multiplications.

The multiplications that define the Gerstenhaber algebra are just Lian and Zuckerman's dot product and bracket, which they use for bosonic strings [13]. Let me recall their definitions. The bracket is given by

$$\{, \} : V_{\gamma_1, k}^s(\alpha, m) \times V_{\gamma_2, l}^s(\beta, n) \rightarrow V_{\gamma_1 + \gamma_2, k+l}^s(\alpha + \beta, m + n - 1) \quad (5.9)$$

$$\{u, v\} = (-1)^a \text{Res}_w \text{Res}_{z-w}(b(z-w)u)(w)v \quad (5.10)$$

with $u \in V^a$. The dot product is

$$\cdot : V_{\gamma_1, k}^s(\alpha, m) \times V_{\gamma_2, l}^s(\beta, n) \rightarrow V_{\gamma_1 + \gamma_2, k+l}^s(\alpha + \beta, m + n) \quad (5.11)$$

$$u \cdot v = \text{Res}_z \frac{u(z)v}{z} \quad (5.12)$$

Both products are well defined iff $\Delta(\gamma_1, \gamma_2) = 0$. As was proven by Lian and Zuckerman, these products define products on the Q -cohomology, satisfying all definitions of a Gerstenhaber algebra. The \mathbf{Z} gradation in the definition of a Gerstenhaber algebra is given by (ghostnumber minus ghostpicture). The bracket has the nice property that it closes on the relative cohomology. Furthermore, if u and v both are in $\ker b_0$ there is the following relation between the dot product and the bracket

$$\{u, v\} = b_0(u \cdot v) \quad (5.13)$$

The dot product does not close on the relative cohomology. The last ingredient that I need to define the Lie algebra on the physical states is the picture

change operation. This is an isomorphism ϕ on the absolute cohomology relating different ghostpictures

$$\phi : H_{\gamma,k}^{\text{abs}}(\alpha, m) \rightarrow H_{\gamma,k+1}^{\text{abs}}(\alpha, m+1) \quad (5.14)$$

For the construction of this isomorphism I refer to the literature [8], but note that it is compatible with the dot and bracket product

$$\phi(\{u, v\}) = \{\phi(u), v\} \bmod Q \quad (5.15)$$

$$\phi(u \cdot v) = \phi(u) \cdot v \bmod Q \quad (5.16)$$

This thus defines the products on the absolute cohomology modulo ghost-picture equivalence, and in particular it defines a Lie-algebra on the physical states $P = H/\phi$. (Note that representatives of cohomology classes in the NS-sector where fermionic in the vertex algebra. They are now bosons in the Lie algebra, as they should). The Lie-algebra is graded by the lattice $E_{1,1}$

$$P = \bigoplus_{\alpha \in E_{1,1}} P_{\alpha} \quad (5.17)$$

5.2 The massless left-moving algebra.

In this section I give a description of the Lie-algebra on the left-moving massless modes. As is well known [12], such states can be described by their momentum p with $p \cdot p = 0$, and by a polarization vector k in the NS-sector, or by a spinor u in the R-sector. This corresponds to the fact that the massless states in the NS-sector transform in the vector representation of the $SO(9, 1)$ symmetry algebra that acts on the superstring algebra. The massless states in the R-sector transform in the spinor representation. I will denote these states by (k, p) and (u, p) respectively. In ghostpicture 0, the state (k, p) is represented by

$$(c(z)k \cdot \partial X(z) + c(z)(p \cdot b(z))(k \cdot b(z)) + \gamma(z)k \cdot b(z)) e^{p \cdot X} \quad (5.18)$$

The polarization vector must satisfy $p \cdot k = 0$. The same state is represented in ghostpicture -1 by

$$c(z)(k \cdot b(z)) e^{p \cdot X} e^{-\phi} \quad (5.19)$$

The commutator

$$[Q, e^{p \cdot X}] = \left(\frac{(p \cdot p) \partial c(z)}{2} + c(z) p \cdot \partial X(z) + \gamma(z) p \cdot b(z) \right) e^{p \cdot X} \quad (5.20)$$

shows that $k \sim k + \alpha p$ with $\alpha \in \mathbf{R}$ in Q -cohomology. The state (u, p) is represented in ghostpicture $-\frac{1}{2}$ by

$$c(z) u^\alpha S_\alpha(z) e^{p \cdot \phi} e^{-\phi/2} \quad (5.21)$$

The spinor u must satisfy the Dirac-equation $\hat{p}u = 0$. Here $\hat{p} = p_\mu \gamma^\mu$, and the γ^μ are Dirac matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. I can now calculate the bracket on the massless states by plugging these representatives into the definition of the bracket. I will not go into detail, but only state the results. Note that the bracket of two states with momenta p_1, p_2 only gives a massless state iff $(p_1 + p_2)^2 = 0$, i.e. $p_1 \cdot p_2 = 0$. Under this condition one finds

$$[(k_1, p_1), (k_2, p_2)] = ((k_1 \cdot p_2)k_2 - (k_2 \cdot p_1)k_1 + (k_1 \cdot k_2)p_1, p_1 + p_2) \quad (5.22)$$

At first sight, this bracket appears not to be antisymmetric. However it is antisymmetric modulo states of the form (p, p) , which are Q -exact.

$$[(k_1, p_1), (u_2, p_2)] = ((k_1 \cdot p_2)u_2 + \frac{1}{2}\hat{p}_1 \hat{k}_1 u_2, p_1 + p_2) \quad (5.23)$$

$$[(u_1, p_1), (u_2, p_2)] = (u_1 \gamma u_2, p_1 + p_2) \quad (5.24)$$

One can check that the bracket is indeed a Lie-bracket, (as far as it closes on massless states), where the (k, p) are bosonic, and the (u, p) are fermionic.

6 On GKM-algebras.

6.1 Chiral algebras.

In this section, I consider some possibilities to construct Lie-algebras. The basic example is just the Lie-algebra on the physical states P (the relative cohomology modulo ghostpicture equivalence), of the chiral (left-moving) string, as described in the previous section. This Lie-algebra is graded by the lattice $E_{1,1}$ (it is actually graded by the whole compactification lattice

E_{10}) and it is the super equivalent of the construction of Borchers [18]. It is therefore a GKM-algebra. Note that the superdimensions of the homogeneous spaces P_α vanish unless $\alpha = 0$

$$\dim P_\alpha^0 - \dim P_\alpha^1 = \begin{cases} 0 & \alpha \neq 0 \\ -6 & \alpha = 0 \end{cases} \quad (6.1)$$

Other $E_{1,1}$ -graded GKM-algebras can be made by replacing the transverse algebra by a more general algebra. This algebra must be a $\mathbf{Z}_2 \times \mathbf{Z}_2$ -graded vertex algebra in order to make the total GSO-projected algebra a vertex superalgebra. The most interesting case is of course when the resulting theory has spacetime supersymmetry. As is well known [9], this requires a transverse algebra with $N = 2$ supersymmetry. This $N = 2$ algebra must have central charge $c = 12$ (or $\hat{c} = 4$), to have $Q^2 = 0$. This replacement leads to a family of $E_{1,1}$ -graded GKM-algebras, whose (graded) dimensions can be expressed in terms of the supersymmetric index $I(q, y)$ of the transverse algebra (for $\alpha \neq 0$)

$$\dim P_\alpha^0 - \dim P_\alpha^1 = -\text{Res}_q q^{\frac{\langle \alpha, \alpha \rangle}{2} - 1} I(q, y = 1) \quad (6.2)$$

By virtue of supersymmetry, these dimensions are zero, unless $\langle \alpha, \alpha \rangle = 0$. Note that the y -dependence of supersymmetric index does not play a role in this Lie-algebra, there is not a gradation of the GKM-algebra corresponding to the J_0 gradation of the transverse algebra.

6.2 BPS-algebras.

In this section I consider the algebra structure on perturbative BPS-states that exists in a type II superstring where the space-dimensions are compactified on $S_1 \times C_4$, and C_4 is some Calabi-Yau fourfold. This construction is analogous to a construction in [15]. A general vertex operator in this theory can be written as

$$e^{k_L \cdot X_L(z)} e^{k_R \cdot X_R(\bar{z})} V(z, \bar{z}) \quad (6.3)$$

where

$$k_L = (E, P_L), \quad k_R = (E, P_R) \quad (6.4)$$

and P_L, P_R are the left- and right moving momenta in the S_1 direction. They are parametrized by the Narain-lattice $\Gamma^{1,1}$, and can be expressed in terms

of winding and momentum numbers m, n as follows

$$P_L = \frac{m}{R} - \frac{nR}{2}, \quad P_R = \frac{m}{R} + \frac{nR}{2} \quad (6.5)$$

Level matching implies that V has left and right conformal dimensions (including ghosts) given by $(n_L, n_R) = (\frac{1}{2}k_L^2, \frac{1}{2}k_R^2)$. The BPS-condition is simply $n_R = 0$, implying $E^2 = P_R^2$, and then

$$n_L = \frac{1}{2}k_L^2 = \frac{1}{2}(P_R^2 - P_L^2) = mn \quad (6.6)$$

I choose the solution $E = P_R$, leading to the BPS-states. (The other solution gives anti-BPS states). The product of two BPS-states is defined in [15] as

$$\mathcal{R}(V_1, V_2) = P(\lim_{z_1 \rightarrow z_2} V(z_1, \bar{z}_1)V_2(z_2, \bar{z}_2) \bmod Q) \quad (6.7)$$

where P projects the resulting vertex operator to the appropriate ghost-numbers (using the operator $b_0 \otimes \bar{b}_0$). In the case that the vertex-operators can be factorized into left- and right moving components (up to a cocycle factor), say

$$V_k(z, \bar{z}) = V_k(z) \otimes \bar{V}(\bar{z})\epsilon_k \quad (6.8)$$

then using (5.13) it follows that the product can be written as

$$\mathcal{R}(V_1, V_2) = \{V_1, V_2\} \otimes \{\bar{V}_1, \bar{V}_2\}\epsilon_1\epsilon_2 \quad (6.9)$$

So it has the structure of a tensor product of two Lie-algebras, and the algebra on the left-movers separately is a Lie-algebra, which by the BPS-condition, is graded by $\Gamma^{1,1}$. Note that this product closes on the space of BPS-states, without the use of Lorentz-boosts. This is by virtue of the fact that the right moving momenta of all BPS-states have the same (lightlike) direction, with a spatial component only in the S_1 direction, and none in the C_4 direction (by assumption). This is so because momentum-operators are ill-defined on a general C_4 . If, however, the Calabi-Yau C_4 is such that momentum-operators can be defined, (for example on a torus) then my definition of the BPS-algebra defines a subalgebra of the usual BPS-algebra, a subalgebra with a specific choice for the right-moving momentum. The entire algebra in this case can only close using Lorentz-boosts of [15].

In the case of toroidal compactification, the algebra on the left-moving parts of the BPS-vertex operators (with given right-moving momentum direction) is just the GKM-algebra as considered in the previous section. It is a subalgebra of the type II equivalent of $\mathcal{H}_0^{\text{mult}}$ in [15].

6.3 Half-twisted model.

In this section I define a Lie-algebra, associated to a Calabi-Yau fourfold C_4 , which has the same graded dimensions as the BPS-algebra defined in the previous section. This construction is based on a so called half-twisted model of [17]. The claim is that the non-linear sigma model associated to C_4 , with the right-moving modes restricted to the NS-sector, has the structure of a $\mathbf{Z}_2 \times \mathbf{Z}_2$ -graded vertex algebra modulo \bar{Q} , where \bar{Q} is the twisted right-moving BRST-operator, equal to $\bar{G}_{-\frac{1}{2}}^+$. Indeed, the OPE

$$V_1(z_1, \bar{z})V_2(z_2, \bar{z}) = \sum_n \frac{W_n(z_2, \bar{z})}{(z_1 - z_2)^n} \text{ mod } \bar{Q} \quad (6.10)$$

is well defined, since there are no poles in \bar{z} [16]. In the case that the vertex-operators can be factorized into left- and right moving components (again up to a cocycle factor), it is not hard to see that this vertex algebra is the tensor product of the vertex algebra on the left-moving modes, and the associative algebra formed by the right moving chiral-ring. The supersymmetric index of this product vertex algebra is equal to the elliptic genus of C_4 . If this algebra is used as transverse algebra in the construction as was described in section 6.1, the result is a unique Lie-algebra associated to the Calabi-Yau fourfold. In the factorizable case, this Lie-algebra is the tensor product of a Lie-algebra on left-moving modes with an associative algebra on the right moving modes.

7 Orbifold constructions.

In this section I will sketch the orbifold compactification of the superstring on $E_{1,1} \otimes (E_8/\mathbf{Z}_2)$. The vertex-algebra is constructed by considering a \mathbf{Z}_2 -twisted construction of the transverse part of the algebra, very much like the construction of the monster vertex algebra [4]. The orbifold is defined by $\theta : V^s \rightarrow V^s$, $\theta^2 = 1$, which acts as the identity on V_γ^L , and

$$\theta V(D_4^*)_\gamma \rightarrow \begin{cases} V(D_4^*)_\gamma & \gamma = 00, 11 \\ -V(D_4^*)_\gamma & \gamma = 10, 01 \end{cases} \quad (7.1)$$

$$\theta V(E_8) \rightarrow V(-E_8) \quad (7.2)$$

Note that θ also commutes with Q . So the cohomology can be split into the positive and negative eigenspaces of θ . I'm interested in the dimensions of these eigenspaces. Since the dimensions are described in terms of the dimensions of the transverse algebra, I need to know how this space splits. The relevant formulas for $V(E_8)$ can be found in [4]. It splits into spaces with partition functions

$$B^\pm = \frac{1}{2} \left(\frac{E_4(q)}{\eta^8(q)} \pm \frac{\eta^8(q)}{\eta^8(q^2)} \right) \quad (7.3)$$

And so the spaces V_{01}^T and V_{11}^T , relevant for the cohomology after GSO-projection, split into spaces with (extended) partition functions

$$\text{Tr}_{(V_\gamma^T)^\pm} q^{L_0-1/2} y^{J_0} = \begin{cases} B^-\theta_2 & \gamma = 01, (+) \\ B^+\theta_2 & \gamma = 01, (-) \\ B^+\theta_1^- & \gamma = 11, (+) \\ B^-\theta_1^- & \gamma = 11, (-) \end{cases} \quad (7.4)$$

And the supersymmetric index of the untwisted transverse part splits into

$$I^+ = B^+\theta_1^- - B^-\theta_2 \quad (7.5)$$

$$I^- = B^-\theta_1^- - B^+\theta_2 \quad (7.6)$$

Next, the twisted transverse part can be constructed out of twisted versions of $V(E_8)$ and $V(D_4^*)$. However, twisting of $V(D_4^*)$ (which itself can be considered as the direct sum of $V(\mathbf{Z}^4)$ and a θ -twisted module of it [10]), gives back the space $V(D_4^*)$. The twisting of $V(E_8)$ gives rise to a space $V^T(E_8)$, and the action of θ can be extended in a natural way to this space such that it splits into spaces with partition functions (see again [4])

$$B_T^\pm = 8 \left(\frac{\eta^8(q)}{\eta^8(\sqrt{q})} \pm \frac{\eta^8(q^2)\eta^8(\sqrt{q})}{\eta^{16}(q)} \right) \quad (7.7)$$

The factor 8 comes from the fact that the lattice algebra of E_8 in the twisted sector has a unique 16-dimensional representation. Note the strange identity $B^- = B_T^-$. So the twisted sector in the orbifold construction is made by replacing the transverse algebra with a twisted algebra as follows

$$V(E_8) \otimes V_\gamma \rightarrow V^T(E_8) \otimes V_{\gamma+11} \quad (7.8)$$

(where the V_γ are part of $V(D_4^*)$). The twisted components relevant for the cohomology in the twisted sector, call them $(V_{01}^T)_T$ and $(V_{11}^T)_T$, split under the extended action of θ , and

$$\text{Tr}_{(V_\gamma^T)_T^\pm} q^{L_0-1/2} y^{J_0} = \begin{cases} B_T^- \theta_2 & \gamma = 01, (+) \\ B_T^+ \theta_2 & \gamma = 01, (-) \\ B_T^+ \theta_1^+ & \gamma = 11, (+) \\ B_T^- \theta_1^+ & \gamma = 11, (-) \end{cases} \quad (7.9)$$

with related supersymmetric indices for the twisted sector

$$I_T^+ = B_T^+ \theta_1^+ - B_T^- \theta_2 \quad (7.10)$$

$$I_T^- = B_T^- \theta_1^+ - B_T^+ \theta_2 \quad (7.11)$$

The entire GSO-projected orbifold-algebra falls apart into four spaces, which I will denote by

$$W_{00} = (V_{00} \oplus V_{10})^+, \quad W_{01} = (V_{00} \oplus V_{10})^- \quad (7.12)$$

$$W_{10} = (V_{00} \oplus V_{10})_T^+, \quad W_{11} = (V_{00} \oplus V_{10})_T^-$$

This algebra is very much like the algebra structure described by Huang in [11]. It's a $\mathbf{Z}_2 \times \mathbf{Z}_2$ -graded vertex algebra. In particular, the space $W_{00} \oplus W_{10}$ can be given the structure of a vertex superalgebra. This algebra corresponds to a chiral orbifold of the superstring. Its construction is very similar to the construction of the monster vertex-algebra [4]. Note that the supersymmetric index associated with the transverse algebra has the properties of a weak Jacobi form of index 2 and weight 0, and actually

$$I^+ + I_T^+ = \frac{1}{6}(5J_1 + J_2) = (y^{-2} + 22 + y^2) + O(q) \quad (7.13)$$

(By the way, the function $I_T^+ - I^- = \frac{1}{6}(J_2 - J_1)$ is proportional to the function that defines the denominator formula for a generalized Kac-Moody algebra which is an automorphic form correction to the Kac-Moody algebra defined by the symmetrized generalized Cartan matrix

$$G_2 = \begin{pmatrix} 4 & -4 & -12 & -4 \\ -4 & 4 & -4 & -12 \\ -12 & -4 & 4 & -4 \\ -4 & -12 & -4 & 4 \end{pmatrix} \quad (7.14)$$

as was found in [20]. This might suggest relations between this chiral orbifold construction and their GKM-algebra.) Instead of looking at a chiral orbifold, I can look at the orbifold of the closed superstring. As is well known, this one is described in terms of the space

$$\bigoplus_{\gamma \in \mathbf{Z}_2 \times \mathbf{Z}_2} W_\gamma^L \oplus W_\gamma^R \quad (7.15)$$

where $W_\gamma^{L,R}$ are two copies (left- and right-moving) of W_γ . Now again I replace the right-moving algebra by the associative algebra of supersymmetric groundstates C_γ , which like W_γ is $\mathbf{Z}_2 \times \mathbf{Z}_2$ -graded. Note that C_{11} has dimension zero, and

$$\begin{aligned} & 8I^+(q, y) - 8I^-(q, y) + 16I_T^+(q, y) + 0I_T^-(q, y) \\ &= \frac{1}{3}(16J_1 + 8J_2) = (8y^{-2} + 32y^{-1} + 304 + 32y + 8y^2) + O(q) \end{aligned} \quad (7.16)$$

correctly reproduces the elliptic genus of the orbifold T_8/\mathbf{Z}_2 , which is a singular manifold with orbifold Euler number 384. The prefactors are of course the superdimensions of the respective C_γ .

Acknowledgement.

I would like to thank R. Dijkgraaf for helpful discussions.

References

- [1] M. Eichler, D. Zagier, *The Theory of Jacobi Forms*, Birkhäuser, Boston, Basel, Stuttgart, 1985
- [2] D. Mumford, *Tata Lectures on Theta*, Birkhäuser, Boston, Basel, Stuttgart, 1983
- [3] C.D.D. Neumann, *The elliptic genus of Calabi-Yau 3- and 4-folds, product formulae and generalized Kac-Moody algebras*, hep-th/9607029
- [4] I. Frenkel, J. Lepowsky, A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Applied Mathematics, Vol. 134, Academic Press, Boston, 1988.

- [5] J. Conway, N. Sloane, *Sphere Packings, Lattices and Groups*, Springer-Verlag, New York, 1988.
- [6] F. ten Kroode, J. van de Leur, *Bosonic and Fermionic realizations of the affine Lie algebra \hat{gl}_n* , Commun.Math.Phys.137:67,1991
- [7] S. Cecotti, P. Fendley, K. Intriligator, C. Vafa, *A New Supersymmetric Index*, Nucl. Phys. B386 (1992) 405-452
- [8] D. Friedan, E. Martinec, S. Shenker, *Conformal invariance, Supersymmetry and String Theory*, Nucl.Phys.B271:93,1986
- [9] T. Banks, L. Dixon, D. Friedan, E. Martinec, *Phenomenology and Conformal field theory or Can String theory predict the weak mixing angle?* Nucl.Phys.B299:613-626,1988
- [10] A. Feingold, I. Frenkel, J. Ries, *Spinor Construction of Vertex Operator Algebras, Triality, and $E_8^{(1)}$* , Contemporary mathematics 121, AMS
- [11] L. Dolan, *Superstring Twisted Conformal field Theory*, Y. Huang, *Nonmeromorphic Extensions of the Moonshine Module*, H. Li, *Local systems of twisted vertex operators, Vertex operator superalgebras and twisted modules*, in *Moonshine, the Monster and Related Topics*, C. Dong, G. Mason, editors, Contemporary Mathematics 193, AMS
- [12] M. Green, J. Schwarz, E. Witten, *Superstring theory Vol. 1 and 2*, Cambridge Monographs on Mathematical Physics.
- [13] B. Lian, G. Zuckerman, *New Perspectives on the BRST-algebraic Structure of String Theory*, Commun.Math.Phys.154:613-646,1993, hep-th/9211072
- [14] B. Lian, G. Zuckerman, *BRST Cohomology of the Super-Virasoro Algebras*, Commun.Math.Phys.125:301-335,1989
- [15] J. Harvey, G. Moore, *On the algebras of BPS states*, hep-th/9609017
- [16] W. Lerche, C. Vafa, N. Warner, *Chiral rings in $N = 2$ superconformal theories*, Nucl.Phys.B324:427,1989

- [17] E. Witten, *Mirror Manifolds and Topological Field Theory*, in *Essays on Mirror Manifolds*, S-T. Yau editor, International Press.
- [18] R. Borcherds, *Monstrous Moonshine and Monstrous Lie superalgebras*, Invent.math.109:405-444,1992
- [19] R. Borcherds, *Vertex algebras, Kac-Moody algebras, and the monster*, Proc.Natl.Acad.Sci.USA 83:3068-3071,1986
- [20] V.A. Gritsenko, V.V. Nikulin, *Siegel Automorphic form corrections of some Lorentzian Kac-Moody Lie algebras*, alg-geom/9504006